Divergence-free virtual element method for Stokes and Navier-Stokes problems

F.Dassi

Department of Mathematics and Applications
University of Milano - Bicocca

a joint work with

L. Beirão da Veiga, A. Russo and G. Vacca

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Montpelier - France
June 19th - 21th 2024

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Divergence-free virtual element method

for Stokes and Navier-Stokes problems

Divergence-free condition

- standard Finite Element do not have it!
- pressure-robust method *only* in a weak sense, for instance N. Ahmed *et al.* (2018).

$$||\nabla(\boldsymbol{u}-\boldsymbol{u}_h)||_{L^2(\Omega)} \lesssim \inf_{\boldsymbol{w}_h\in\mathbf{V}_h}||\nabla(\boldsymbol{u}-\boldsymbol{w}_h)||_{L^2(\Omega)} + \frac{1}{\nu}\inf_{q_h\in Q_h}||p-q_h||_{L^2(\Omega)}$$

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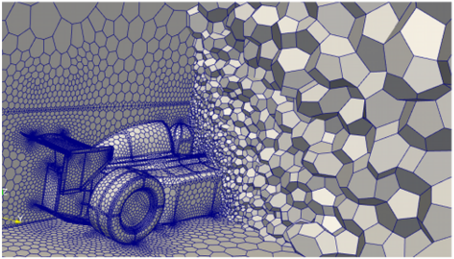
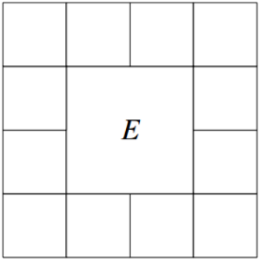
Divergence-free **virtual element method**

for Stokes and Navier-Stokes problems

What is the Virtual Element Method (VEM)?

A generalization of the Finite Element Method introduced in 2013





- general **polygonal** and **polyhedral** meshes (also non convex)
- additional interesting **features** and **properties**

"Basic principles of virtual element method"
L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini and A. Russo (2013)

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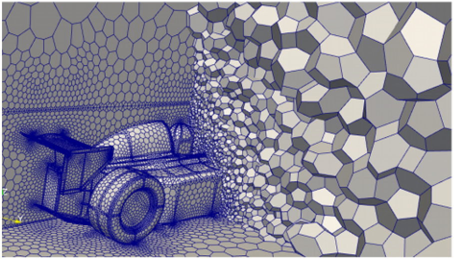

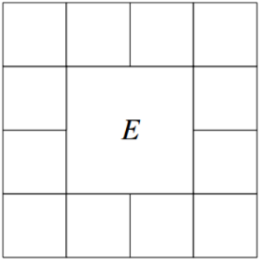

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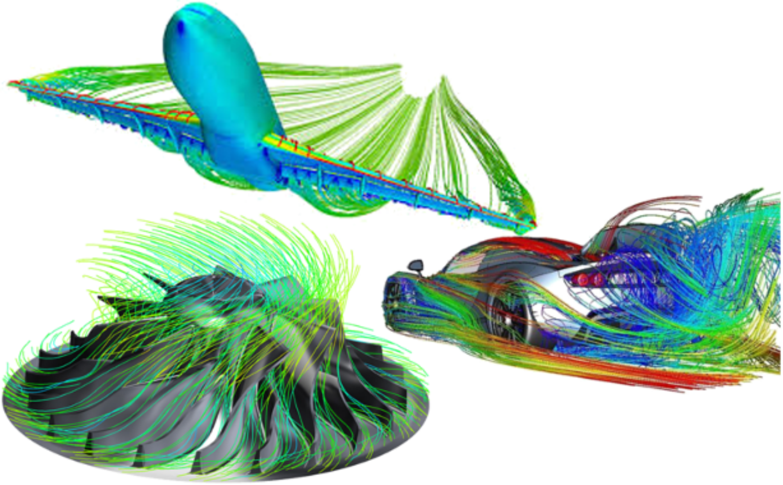
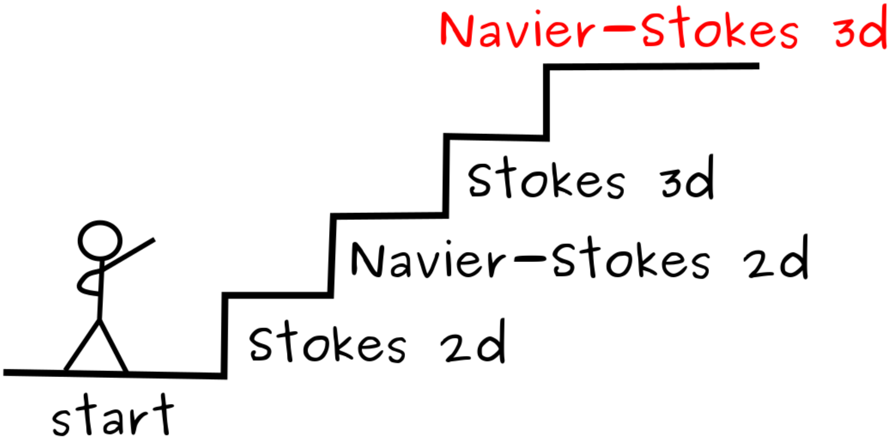
Divergence-free virtual element method

for Stokes and Navier-Stokes problems

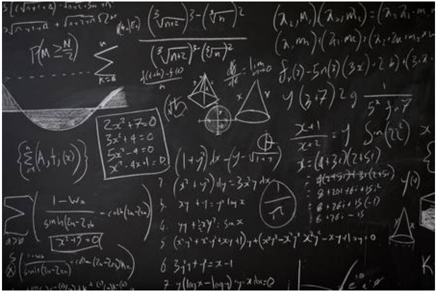
Stokes and Navier-Stokes problems




Navier-Stokes 3d
Stokes 3d
Navier-Stokes 2d
Stokes 2d
start

Talk outline

Problem definition



Stokes problem - continuous formulation

We search for a velocity field $\boldsymbol{u}$ and pressure p, such that

$$\begin{cases} -\nu\,\boldsymbol{\Delta}\boldsymbol{u} - \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \mathrm{div}(\boldsymbol{u}) &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where

- Ω be a simply connected domain in $\mathbb{R}^2$
- $\mathbf{f} \in [L^2(\Omega)]^2$

Stokes problem - variational formulation

$$
\begin{cases}
\text{find } (\boldsymbol{u},\, p) \in \mathbf{V}_0(\Omega) \times Q(\Omega) \text{ such that:} \\
\displaystyle\int_\Omega \nu\, \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\, \mathrm{d}\Omega + \int_\Omega \mathrm{div}(\boldsymbol{v})\, p\, \mathrm{d}\Omega = \int_\Omega \mathbf{f} \cdot \boldsymbol{v}\, \mathrm{d}\Omega & \forall \boldsymbol{v} \in \mathbf{V}_0(\Omega) \\
\displaystyle\int_\Omega \mathrm{div}(\boldsymbol{u})\, q\, \mathrm{d}\Omega = 0 & \forall q \in Q(\Omega)
\end{cases}
$$

$$
\begin{aligned}
\mathbf{V}_0(\Omega) &:= \left\{\boldsymbol{v} \in [H^1(\Omega)]^2 \,:\, \boldsymbol{v} = 0 \text{ on } \partial\Omega\right\} \\
Q(\Omega) &:= \left\{q \in L^2(\Omega) \,:\, \int_\Omega q\, \mathrm{d}\Omega = 0\right\}
\end{aligned}
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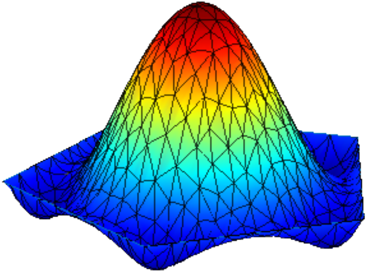
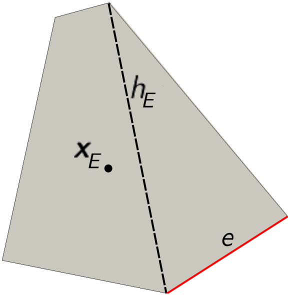
VEM spaces



Notation for polygons

polygon E



$|E| = \text{area}$

Notation for 2d monomials

Let $E \subset \mathbb{R}^2$, $k \in \mathbb{N}\backslash\{0\}$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ be a multi-index, we define the **scaled monomials**

$$m_{\boldsymbol{\alpha}} := \left(\frac{x - x_E}{h_E}\right)^{\alpha_1} \left(\frac{y - y_E}{h_E}\right)^{\alpha_2}$$

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$$\boldsymbol{m}^1_{\boldsymbol{\alpha}} = \begin{bmatrix} m_{\boldsymbol{\alpha}} \\ 0 \end{bmatrix}, \quad \boldsymbol{m}^2_{\boldsymbol{\alpha}} = \begin{bmatrix} 0 \\ m_{\boldsymbol{\alpha}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{m}^\perp = \begin{bmatrix} m_{(0,1)} \\ -m_{(1,0)} \end{bmatrix}$$

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- $\{\boldsymbol{m}^i_{\boldsymbol{\alpha}}\}$ is a basis of $[\mathbb{P}_k(E)]^2$

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3d case is analogous

Polynomial decomposition

It is **essential** the following property

$$[\mathbb{P}_k(E)]^2 = \nabla\mathbb{P}_{k+1}(E) \oplus \mathbf{x}^\perp\mathbb{P}_{k-1}(E)$$

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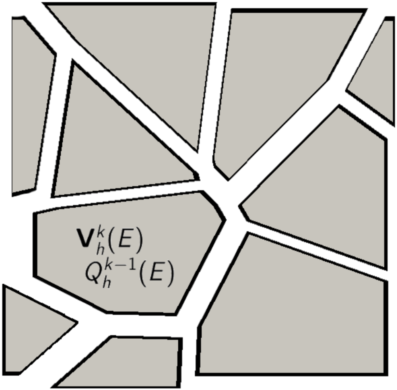
We found a recipe for $\boldsymbol{m}_{\alpha}^{i}$!!

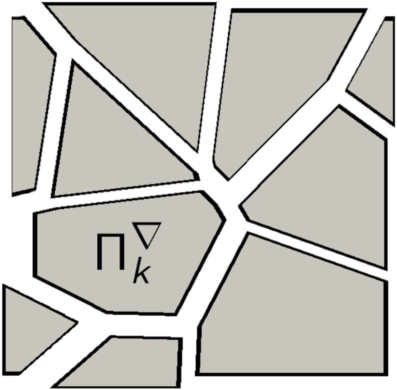
VEM space definition - the plan

- VEM local spaces



VEM space definition - the plan

- VEM local spaces
- local projection operators



VEM space definition - the plan

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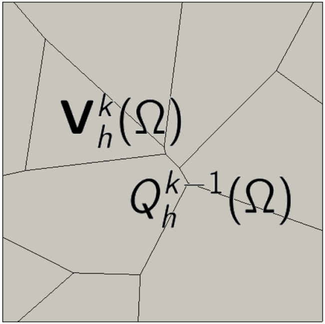
- glue spaces



Velocity field virtual space

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\mathbf{V}_h^k(E) := \Big\{ \boldsymbol{v}_h \in [H^1(E) \cap C^0(E)]^2 \,:\, \boldsymbol{v}_h|_e \in [\mathbb{P}_k(e)]^2 \; \forall e \in \partial E \,, \\
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Velocity field virtual space - d.o.f.

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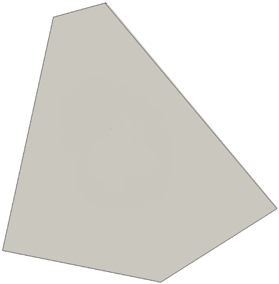

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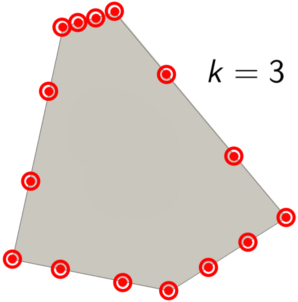

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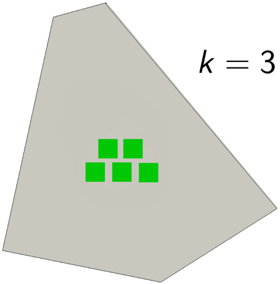
- $k(k+1)/2 - 1$ divergence moments

$$\int_E \mathrm{div}(\boldsymbol{v}_h)\, m_\alpha \, \mathrm{d}E$$



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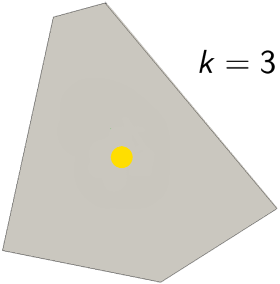
- $(k-1)(k-2)/2$ perp moments

$$\int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp)\, m_{\boldsymbol{\beta}} \,\mathrm{d}E$$



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- vectorial values at the ...
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  $\int_E \ldots$

- $(k-1)(k-\ldots$

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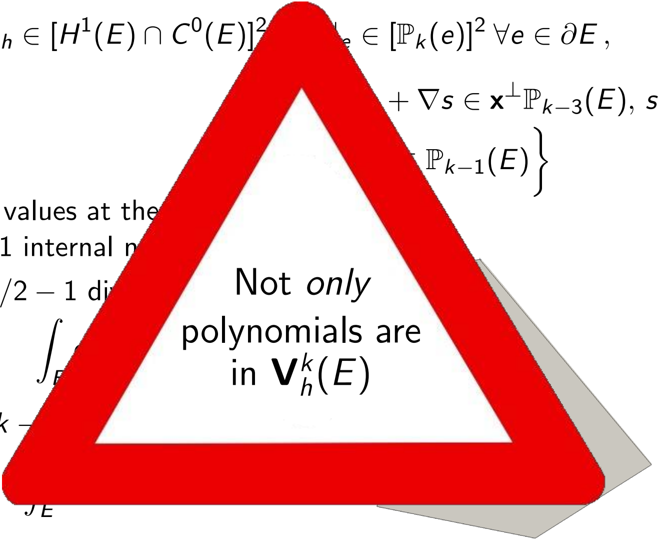

Projection operator Π_k^{∇}

$$
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\begin{array}{rcl}
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- $\mathbb{M}_k(E) := \{\boldsymbol{m}_i\}_{i=1}^n$
- $\mathbb{M}_0(E) := \{\boldsymbol{m}_1,\, \boldsymbol{m}_2\}$

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- $\mathbb{M}_0(E) := \{\boldsymbol{m}_1,\, \boldsymbol{m}_2\}$
- $\Pi_k^\nabla \boldsymbol{v}_h := c_0\boldsymbol{m}_1 + c_1\boldsymbol{m}_1 + \ldots + c_n\boldsymbol{m}_n$

Projection operator Π_k^∇

$$
\begin{cases}
\displaystyle\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla} \boldsymbol{p}_k \,\mathrm{d}E = 0 & \forall \boldsymbol{p}_k \in [\mathbb{P}_k(E)]^2 \\
\displaystyle\int_{\partial E} (\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) \cdot \boldsymbol{p}_0 \,\mathrm{d}e = 0 & \forall \boldsymbol{p}_0 \in [\mathbb{P}_0(E)]^2
\end{cases}
$$

- $\mathbb{M}_k(E) := \{\boldsymbol{m}_i\}_{i=1}^n$
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- $\Pi_k^\nabla \boldsymbol{v}_h := c_0 \boldsymbol{m}_1 + c_1 \boldsymbol{m}_1 + \ldots + c_n \boldsymbol{m}_n$
- conditions according to $\mathbb{M}_k(E)$ and $\mathbb{M}_0(E)$

Projection operator Π_k^∇

$$
\begin{cases}
\displaystyle\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h)\, \boldsymbol{\nabla} \boldsymbol{p}_k \,\mathrm{d}E = 0 & \forall \boldsymbol{p}_k \in [\mathbb{P}_k(E)]^2 \\
\displaystyle\int_{\partial E} (\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) \cdot \boldsymbol{p}_0 \,\mathrm{d}e = 0 & \forall \boldsymbol{p}_0 \in [\mathbb{P}_0(E)]^2
\end{cases}
$$

- $\mathbb{M}_k(E) := \{\boldsymbol{m}_i\}_{i=1}^n$
- $\mathbb{M}_0(E) := \{\boldsymbol{m}_1,\, \boldsymbol{m}_2\}$
- $\Pi_k^\nabla \boldsymbol{v}_h := c_0 \boldsymbol{m}_1 + c_1 \boldsymbol{m}_1 + \ldots +$
- conditions according to $\mathbb{M}_k(E)$ a

Projection operator Π_k^∇ - Yes we can compute it!

$$\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla} \boldsymbol{m}_i \,\mathrm{d}E = 0$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$\begin{aligned} \int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= 0 \\ \sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E \end{aligned}$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$\begin{aligned}\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= 0\\ \sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E\end{aligned}$$

Focus on the virtual part

$$\int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$\begin{aligned}\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i\,\mathrm{d}E &= 0\\ \sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i\,\mathrm{d}E &= \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i\,\mathrm{d}E\end{aligned}$$

Focus on the virtual part

$$\int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i\,\mathrm{d}E = -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i\,\mathrm{d}E + \int_{\partial E} \boldsymbol{v}_h \cdot (\boldsymbol{\nabla}\boldsymbol{m}_i\,\mathbf{n})\,\mathrm{d}e$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= 0 \\
\sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E
\end{aligned}
$$

Focus on the virtual part

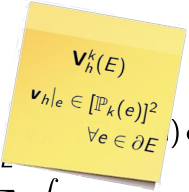
$$
\begin{aligned}
\int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \int_{\partial E} \boldsymbol{v}_h \cdot (\boldsymbol{\nabla}\boldsymbol{m}_i\,\mathbf{n})\,\mathrm{d}e \\
&= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \sum_{e\in\partial E} \int_e \boldsymbol{v}_h \cdot (\boldsymbol{\nabla}\boldsymbol{m}_i\,\mathbf{n}_e)\,\mathrm{d}e
\end{aligned}
$$

Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= 0 \\
\sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla}\boldsymbol{m}_j : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= \int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E
\end{aligned}
$$

Focus on the virtual part

$$
\begin{aligned}
\int_E \boldsymbol{\nabla}\boldsymbol{v}_h : \boldsymbol{\nabla}\boldsymbol{m}_i \,\mathrm{d}E &= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \int_{\partial E} \cdots) \,\mathrm{d}e \\
&= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta}\boldsymbol{m}_i \,\mathrm{d}E + \sum_{e \in \partial E} \int_e \boldsymbol{v}_h \cdot (\boldsymbol{\nabla}\boldsymbol{m}_i \, \mathbf{n}_e) \,\mathrm{d}e
\end{aligned}
$$



Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{\nabla}(\boldsymbol{v}_h - \Pi_k^\nabla \boldsymbol{v}_h) : \boldsymbol{\nabla} \boldsymbol{m}_i \, \mathrm{d}E &= 0 \\
\sum_{j=1}^{n} c_j \int_E \boldsymbol{\nabla} \boldsymbol{m}_j : \boldsymbol{\nabla} \boldsymbol{m}_i \, \mathrm{d}E &= \int_E \boldsymbol{\nabla} \boldsymbol{v}_h : \boldsymbol{\nabla} \boldsymbol{m}_i \, \mathrm{d}E
\end{aligned}
$$

Focus on the virtual part

$$
\begin{aligned}
\int_E \boldsymbol{\nabla} \boldsymbol{v}_h : \boldsymbol{\nabla} \boldsymbol{m}_i \, \mathrm{d}E &= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta} \boldsymbol{m}_i \, \mathrm{d}E + \int_{\partial E} \cdots) \, \mathrm{d}e \\
&= -\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta} \boldsymbol{m}_i \, \mathrm{d}E + \sum_{e \in \partial E} \int_e \boldsymbol{v}_h \cdot (\boldsymbol{\nabla} \boldsymbol{m}_i \, \mathbf{n}_e) \, \mathrm{d}e
\end{aligned}
$$

$\mathbf{V}_h^k(E)$

$\boldsymbol{v}_h|_e \in [\mathbb{P}_k(e)]^2$

$\forall e \in \partial E$

Projection operator Π_k^∇ - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta} \boldsymbol{m}_i \,\mathrm{d}E$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta m}_i \,\mathrm{d}E$$

$$\boldsymbol{\Delta m}_i = c_\alpha \boldsymbol{m}_\alpha + c_\beta \boldsymbol{m}_\beta + c_\delta \boldsymbol{m}_\delta + c_\gamma \boldsymbol{m}_\gamma$$

Projection operator Π_k^∇ - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta} \boldsymbol{m}_i \,\mathrm{d}E$$

$$\boldsymbol{\Delta} \boldsymbol{m}_i = c_\alpha \boldsymbol{m}_\alpha + c_\beta \boldsymbol{m}_\beta + c_\delta \boldsymbol{m}_\delta + c_\gamma \boldsymbol{m}_\gamma$$

$$\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta, \boldsymbol{m}_\delta, \boldsymbol{m}_\gamma \in \mathbb{P}_{k-2}(E)$$

Projection operator Π_k^∇ - Yes we can compute it!

$$-\int_E \boldsymbol{v}_h \cdot \boldsymbol{\Delta m}_i \,\mathrm{d}E$$

$$\boldsymbol{\Delta m}_i = c_{\boldsymbol{\alpha}} \boldsymbol{m}_{\boldsymbol{\alpha}} + c_{\boldsymbol{\beta}} \boldsymbol{m}_{\boldsymbol{\beta}} + c_{\boldsymbol{\delta}} \boldsymbol{m}_{\boldsymbol{\delta}} + c_\gamma \boldsymbol{m}_\gamma$$

$$\boldsymbol{m}_\alpha, \boldsymbol{m}_\beta, \boldsymbol{m}_\delta, \boldsymbol{m}_\gamma \in \mathbb{P}_{k-2}(E)$$

$$\boldsymbol{m}_{\boldsymbol{\alpha}} = c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_\eta$$

$m_\zeta \in \mathbb{P}_{k-1}(E)$ and $m_{\boldsymbol{\eta}} \in \mathbb{P}_{k-3}(E)$

Projection operator Π_k^{∇} - Yes we can compute it!

$$\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_{\boldsymbol{\alpha}} \,\mathrm{d}E \quad = \quad \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_{\boldsymbol{\eta}}) \,\mathrm{d}E$$

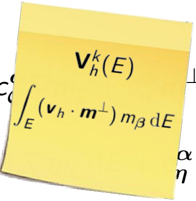
Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_{\boldsymbol{\alpha}} \,\mathrm{d}E &= \int_E \boldsymbol{v}_h \cdot (c_{\zeta}^{\boldsymbol{\alpha}} \nabla m_{\zeta} + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^{\perp} m_{\boldsymbol{\eta}}) \,\mathrm{d}E \\
&= c_{\zeta}^{\boldsymbol{\alpha}} \int_E \boldsymbol{v}_h \cdot \nabla m_{\zeta} \,\mathrm{d}E + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^{\perp}) m_{\boldsymbol{\eta}} \,\mathrm{d}E
\end{aligned}
$$

Projection operator Π_k^∇ - Yes we can compute it!

$$\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E = \int_E \boldsymbol{v}_h \cdot (c_\zeta^\alpha \;\cdots\; {}^\perp m_\eta)\,\mathrm{d}E$$

$$= c_\zeta^\alpha \int_E \boldsymbol{v}_h \;\cdots\; {}_\eta^\alpha \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_\eta \,\mathrm{d}E$$



Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E &= \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_\eta) \,\mathrm{d}E \\
&= c_\zeta^{\boldsymbol{\alpha}} \int_E \boldsymbol{v}_h \cdot \nabla m_\zeta \,\mathrm{d}E + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_{\boldsymbol{\eta}} \,\mathrm{d}E \\
&= -c_\zeta^{\boldsymbol{\alpha}} \int_E \mathrm{div}(\boldsymbol{v}_h) m_\zeta \,\mathrm{d}E + c_\zeta^{\boldsymbol{\alpha}} \int_{\partial E} (\boldsymbol{v}_h \cdot \mathbf{n})\, m_\zeta \,\mathrm{d}e + \ldots
\end{aligned}
$$

Projection operator Π_k^{∇} - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_\alpha \,\mathrm{d}E &= \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_{\boldsymbol{\eta}}) \,\mathrm{d}E \\
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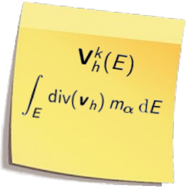
\end{aligned}
$$



Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_{\boldsymbol{\alpha}} \,\mathrm{d}E &= \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_{\boldsymbol{\eta}}) \,\mathrm{d}E \\
&= c_\zeta^{\boldsymbol{\alpha}} \int_E \boldsymbol{v}_h \cdot \nabla m_\zeta \,\mathrm{d}E + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_{\boldsymbol{\eta}} \,\mathrm{d}E \\
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&= -c_\zeta^{\boldsymbol{\alpha}} \int_E \mathrm{div}(\boldsymbol{v}_h) m_\zeta \,\mathrm{d}E + c_\zeta^{\boldsymbol{\alpha}} \sum_{e \in \partial E} (\boldsymbol{v}_h \cdot \mathbf{n}_e)\, m_\zeta \,\mathrm{d}e + \ldots
\end{aligned}
$$

Projection operator Π_k^∇ - Yes we can compute it!

$$
\begin{aligned}
\int_E \boldsymbol{v}_h \cdot \boldsymbol{m}_{\boldsymbol{\alpha}} \,\mathrm{d}E &= \int_E \boldsymbol{v}_h \cdot (c_\zeta^{\boldsymbol{\alpha}} \nabla m_\zeta + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \boldsymbol{m}^\perp m_{\boldsymbol{\eta}}) \,\mathrm{d}E \\
&= c_\zeta^{\boldsymbol{\alpha}} \int_E \boldsymbol{v}_h \cdot \nabla m_\zeta \,\mathrm{d}E + c_{\boldsymbol{\eta}}^{\boldsymbol{\alpha}} \int_E (\boldsymbol{v}_h \cdot \boldsymbol{m}^\perp) m_{\boldsymbol{\eta}} \,\mathrm{d}E \\
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&= -c_\zeta^{\boldsymbol{\alpha}} \int_E \mathrm{div}(\boldsymbol{v}_h) m_\zeta \,\mathrm{d}E + c_\zeta^{\boldsymbol{\alpha}} \sum_{e \in \partial E} (\boldsymbol{v}_h \cdot \mathbf{n}_e)\, m_\zeta \,\mathrm{d}e + \ldots
\end{aligned}
$$

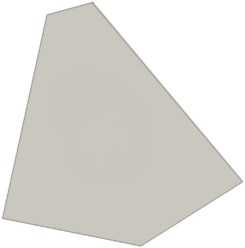
$\mathbf{V}_h^k(E)$

$\boldsymbol{v}_h|_e \in [\mathbb{P}_k(e)]^2$

$\forall e \in \partial E$

Pressure virtual space and d.o.f.

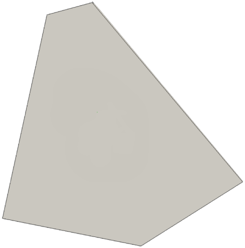
$$Q_h^{k-1}(E) := \{q_h \ : \ q_h \in \mathbb{P}_{k-1}(E)\}$$



Pressure virtual space and d.o.f.

$$Q_h^{k-1}(E) := \{q_h \,:\, q_h \in \mathbb{P}_{k-1}(E)\}$$

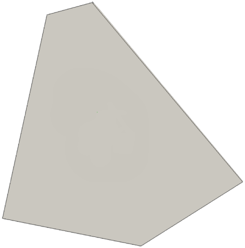
- no VEM approximation



Pressure virtual space and d.o.f.

$$Q_h^{k-1}(E) := \{q_h \,:\, q_h \in \mathbb{P}_{k-1}(E)\}$$

- no VEM approximation
- no projection

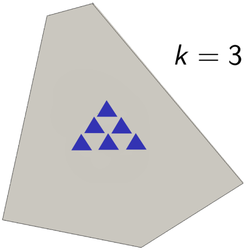


Pressure virtual space and d.o.f.

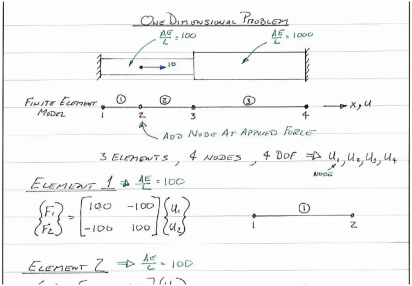
$$Q_h^{k-1}(E) := \{q_h \,:\, q_h \in \mathbb{P}_{k-1}(E)\}$$

- no VEM approximation
- no projection
- $k(k+1)/2$ moments

$$\int_E q_h \, m_\alpha \, \mathrm{d}E$$



Problem discretization



$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Problem discretization

Consider a polyhedral decomposition Ω_h of Ω, then we solve:

$$
\begin{cases}
\text{find } (\boldsymbol{u}_h, p_h) \in \mathbf{V}_h^k \times Q_h^{k-1} \text{ such that} \\
a_h(\boldsymbol{u}_h\,, \boldsymbol{v}_h) + \displaystyle\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \int_{\Omega_h} \mathbf{f}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\Omega_h & \forall \boldsymbol{v}_h \in \mathbf{V}_{h,0}^k \\
\displaystyle\int_{\Omega_h} \mathrm{div}(\boldsymbol{u}_h)\, q_h \,\mathrm{d}\Omega_h = 0 & \forall q_h \in Q_h^{k-1}
\end{cases}
$$

Problem discretization

Consider a polyhedral decomposition Ω_h of Ω, then we solve:

$$\begin{cases} \text{find } (\boldsymbol{u}_h, p_h) \in \mathbf{V}_h^k \times Q_h^{k-1} \text{ such that} \\ a_h(\boldsymbol{u}_h\,, \boldsymbol{v}_h) + \displaystyle\int_{\Omega_h} \operatorname{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \int_{\Omega_h} \mathbf{f}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\Omega_h & \forall \boldsymbol{v}_h \in \mathbf{V}_{h,0}^k \\ \displaystyle\int_{\Omega_h} \operatorname{div}(\boldsymbol{u}_h)\, q_h \,\mathrm{d}\Omega_h = 0 & \forall q_h \in Q_h^{k-1} \end{cases}$$

where

- we define from Π_k^∇ and dofs

$$a_h(\cdot,\cdot) \approx \int_\Omega \nu \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\Omega$$

Problem discretization

Consider a polyhedral decomposition Ω_h of Ω, then we solve:

$$\begin{cases} \text{find } (\boldsymbol{u}_h, p_h) \in \mathbf{V}_h^k \times Q_h^{k-1} \text{ such that} \\ a_h(\boldsymbol{u}_h\,, \boldsymbol{v}_h) + \displaystyle\int_{\Omega_h} \operatorname{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \int_{\Omega_h} \mathbf{f}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\Omega_h & \forall \boldsymbol{v}_h \in \mathbf{V}_{h,0}^k \\ \displaystyle\int_{\Omega_h} \operatorname{div}(\boldsymbol{u}_h)\, q_h \,\mathrm{d}\Omega_h = 0 & \forall q_h \in Q_h^{k-1} \end{cases}$$

where

- we define from Π_k^{∇} and dofs

$$a_h(\cdot,\cdot) \approx \int_\Omega \nu \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\Omega$$

- $\mathbf{f}_h$ is a proper L^2 projection of $\mathbf{f}$

Defintion of $a_h(\cdot, \cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\, \boldsymbol{w}_h) = \sum_{E\in\Omega_h} a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h)$$

Defintion of $a_h(\cdot,\cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\,\boldsymbol{w}_h) = \sum_{E\in\Omega_h} a_{h,E}(\boldsymbol{v}_h,\,\boldsymbol{w}_h)$$

where

$$\begin{aligned} a_{h,E}(\boldsymbol{v}_h,\,\boldsymbol{w}_h) \quad &:= \quad \int_E \boldsymbol{\nabla}(\Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}(\Pi_k^{\nabla}\boldsymbol{w}_h)\,\mathrm{d}E \\ &\qquad + s_E(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h,\,\boldsymbol{w}_h - \Pi_k^{\nabla}\boldsymbol{w}_h) \end{aligned}$$

Defintion of $a_h(\cdot, \cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\, \boldsymbol{w}_h) = \sum_{E \in \Omega_h} a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h)$$

consistency

where

$$a_{h,E}(\boldsymbol{v}_h,\, \boldsymbol{w}_h) \quad := \quad \int_E \boldsymbol{\nabla}(\Pi_k^{\nabla} \boldsymbol{v}_h) : \boldsymbol{\nabla}(\Pi_k^{\nabla} \boldsymbol{w}_h)\, \mathrm{d}E$$
$$+ s_E(\boldsymbol{v}_h - \Pi_k^{\nabla} \boldsymbol{v}_h,\, \boldsymbol{w}_h - \Pi_k^{\nabla} \boldsymbol{w}_h)$$

Defintion of $a_h(\cdot,\cdot)$

Follow a standard VEM approach

$$a_h(\boldsymbol{v}_h,\,\boldsymbol{w}_h) = \sum_{E\in\Omega_h} a_{h,E}(\boldsymbol{v}_h,\,\boldsymbol{w}_h)$$

where

$$a_{h,E}(\boldsymbol{v}_h,\,\boldsymbol{w}_h) \;:=\; \int_E \boldsymbol{\nabla}(\Pi_k^{\nabla}\boldsymbol{v}_h) : \boldsymbol{\nabla}(\Pi_k^{\nabla}\boldsymbol{w}_h)\,\mathrm{d}E$$
$$+ s_E(\boldsymbol{v}_h - \Pi_k^{\nabla}\boldsymbol{v}_h,\,\boldsymbol{w}_h - \Pi_k^{\nabla}\boldsymbol{w}_h)$$

stability

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h$$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

there is no approximation

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

there is no approximation

if $p_h \in \mathbb{R}$

$$\begin{aligned}
\int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E &= p_h \int_E \mathrm{div}(\boldsymbol{v}_h) \,\mathrm{d}E \\
&= p_h \int_{\partial E} (\boldsymbol{v}_h \cdot \mathbf{n}) \,\mathrm{d}e \\
&= p_h \sum_{e\in\partial E} \int_e (\boldsymbol{v}_h \cdot \mathbf{n}_e) \,\mathrm{d}e
\end{aligned}$$

Mixed term

$$\int_{\Omega_h} \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E$$

there is no approximation

if $p_h \in \mathbb{R}$

$$\begin{aligned}\int_E \mathrm{div}(\boldsymbol{v}_h)\, p_h \,\mathrm{d}E &= p_h \int_E \mathrm{div}(\boldsymbol{v}_h)\,\mathrm{d}E \\ &= p_h \int_{\partial E} (\boldsymbol{v}_h \cdot \mathbf{n})\,\mathrm{d}e \\ &= p_h \sum_{e\in\partial E} \int_e (\boldsymbol{v}_h \cdot \mathbf{n}_e)\,\mathrm{d}e\end{aligned}$$

$\mathbf{V}_h^k(E)$

$\boldsymbol{v}_h|_e \in [\mathbb{P}_k(e)]^2$

$\forall e \in \partial E$

Mixed term

$$\int_{\Omega_h} \operatorname{div}(\boldsymbol{v}_h)\, p_h \, \mathrm{d}\Omega_h = \sum_{E\in\Omega_h} \int_E \operatorname{div}(\boldsymbol{v}_h)\, p_h \, \mathrm{d}E$$

there is no approximation

if $p_h \in \mathbb{P}_{k-1}(E)\backslash\mathbb{R}$

$$\int_E \operatorname{div}(\boldsymbol{v}_h)\, p_h \, \mathrm{d}E = \sum_{s=1}^{n} c_s^{p_h} \int_E \operatorname{div}(\boldsymbol{v}_h)\, m_s \, \mathrm{d}E$$

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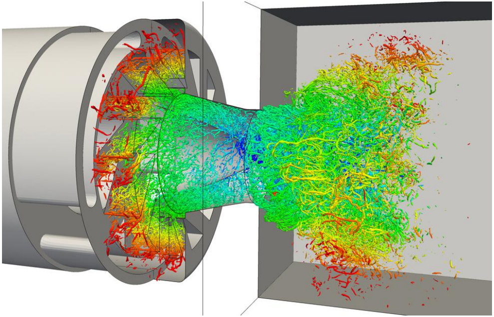
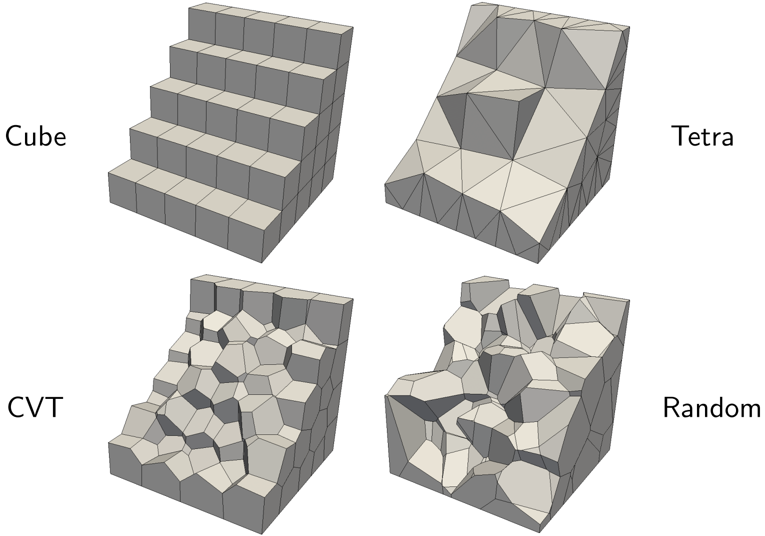
$$\mathbf{V}_h^k(E)$$

$$\int_E \operatorname{div}(\boldsymbol{v}_h)\, m_\alpha \,\mathrm{d}E$$

Numerical examples



Mesh types



Error norms

- H^1**–velocity error**:

$$e^{\boldsymbol{u}}_{H^1} := \sqrt{\sum_{E\in\Omega_h} \left|\left|\nabla \boldsymbol{u} - \boldsymbol{\Pi}^0_{k-1}\, \nabla \boldsymbol{u}_h\right|\right|^2_{L^2(E)}} \sim h^k$$

- L^2**–pressure error**:

$$e^{p}_{L^2} := \sqrt{\sum_{E\in\Omega_h} \left|\left|p - p_h\right|\right|^2_{L^2(E)}} \sim h^k$$

"Divergence free Virtual Elements for the Stokes problem on polygonal meshes"
L. Beirão da Veiga, C. Lovadina, and G.Vacca (2017)

Example 1: Convergence analysis for Stokes

Let us consider a Stokes problem

$$\begin{cases} -\nu\,\boldsymbol{\Delta u} - \nabla p & = & \mathbf{f} & \text{in } \Omega \\ \mathrm{div}(\boldsymbol{u}) & = & 0 & \text{in } \Omega \\ \boldsymbol{u} & = & \boldsymbol{r} & \text{on } \partial\Omega \end{cases}$$

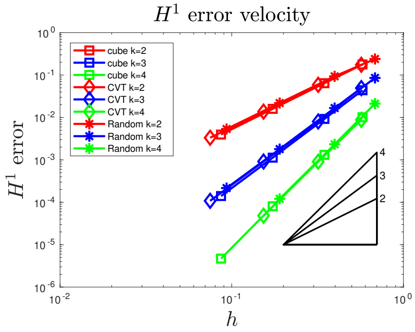
where the exact solution is

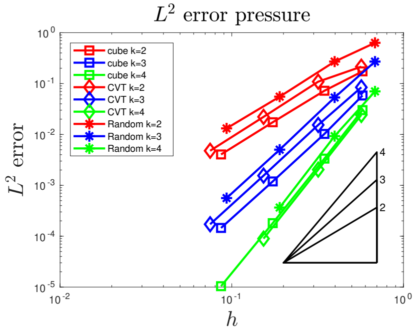
$$\boldsymbol{u}(x,\,y,\,z) := \begin{pmatrix} \sin(\pi x)\,\cos(\pi y)\,\cos(\pi z) \\ \cos(\pi x)\,\sin(\pi y)\,\cos(\pi z) \\ -2\cos(\pi x)\,\cos(\pi y)\,\sin(\pi z) \end{pmatrix}$$

and

$$p(x,\,y,\,z) := -\pi\,\cos(\pi x)\,\cos(\pi y)\,\cos(\pi z)\,.$$

Example 1: Convergence analysis for Stokes





"The Stokes complex for Virtual Elements in three dimensions"
L. Beirão da Veiga, F, Dassi, and G.Vacca submitted

Example 2: Convergence analysis for Navier-Stokes

Let us consider a Navier-Stokes problem

$$\begin{cases} -\nu\, \boldsymbol{\Delta u} + \boldsymbol{u}\nabla\boldsymbol{u} - \nabla p & = & \mathbf{f} & \text{in } \Omega \\ \text{div}(\boldsymbol{u}) & = & 0 & \text{in } \Omega \\ \boldsymbol{u} & = & \boldsymbol{r} & \text{on } \partial\Omega \end{cases}$$

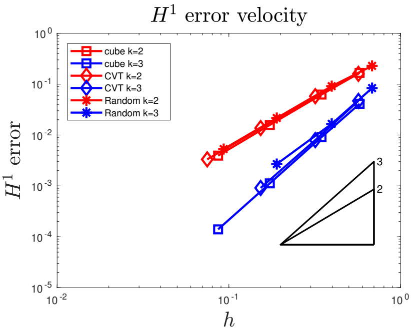
where the exact solution is

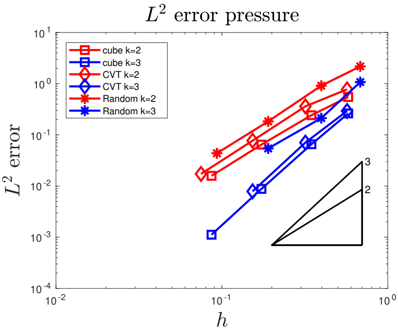
$$\boldsymbol{u}(x,\, y,\, z) := \left(\begin{array}{r} \sin(\pi x)\, \cos(\pi y)\, \cos(\pi z) \\ \cos(\pi x)\, \sin(\pi y)\, \cos(\pi z) \\ -2\cos(\pi x)\, \cos(\pi y)\, \sin(\pi z) \end{array} \right)$$

and

$$p(x,\, y,\, z) := \sin(2\pi x)\, \sin(2\pi y)\, \sin(2\pi z)\,.$$

Example 2: Convergence analysis for Navier-Stokes





"The Stokes complex for Virtual Elements in three dimensions"
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Example 3: Benchmark problems

Let us consider a Stokes problem, we have the following estimate

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 \lesssim h^s \, \mathcal{F}(\boldsymbol{u};\, \nu, \gamma) + h^{s+2} \, \mathcal{H}(\mathbf{f}; \nu)$$

for suitable functions $\mathcal{F}$, $\mathcal{H}$, $\mathcal{K}$ independent of h.

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Example 3: Benchmark problems

We consider two problems

$$\boldsymbol{u}(x,\, y,\, z) := \begin{pmatrix} k\,x\,z^{k-1} \\ k\,y\,z^{k-1} \\ (2-k)\,x^k + (2-k)\,y^k - 2\,z^k \end{pmatrix},$$

and

$$p_1(x,\, y,\, z) := x^k\,y + y^k\,z + z^k\,x - \frac{3}{2(k+1)}\,,$$

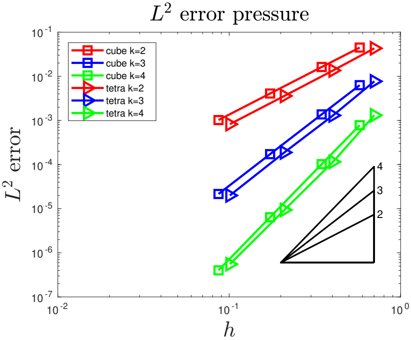
or

$$p_2(x,\, y,\, z) := \sin(2\pi x)\,\sin(2\pi y)\,\sin(2\pi z)\,.$$

Example 3: Benchmark problem, case p_1

H^1 error velocity

k	**Cube**	**Tetra**
2	1.0576e-13	7.2075e-13
3	2.7333e-13	1.1927e-12
4	1.5266e-12	2.2718e-10



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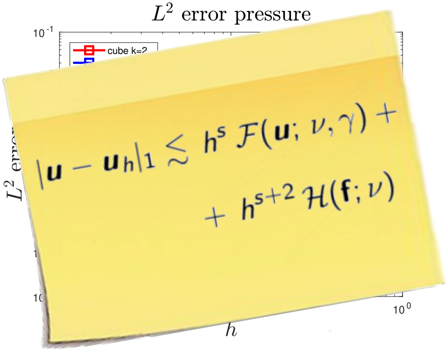

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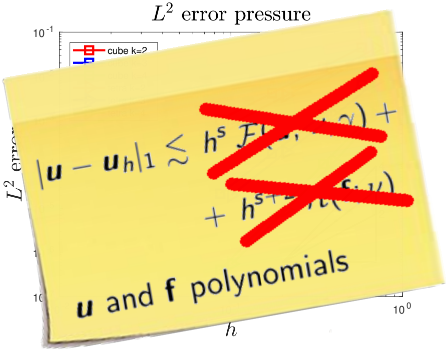

"The Stokes complex for Virtual Elements in three dimensions"
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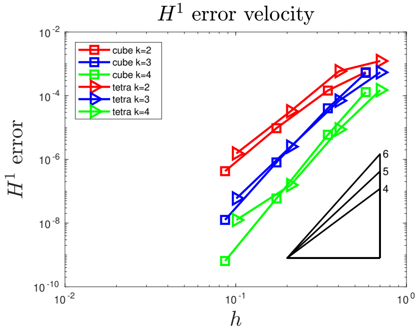
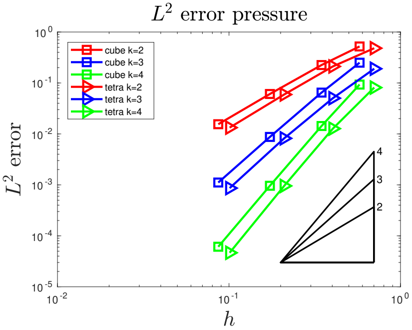
Example 3: Benchmark problem, case p_2





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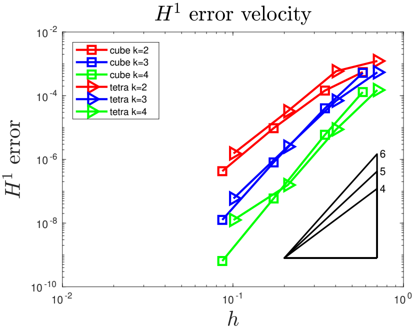
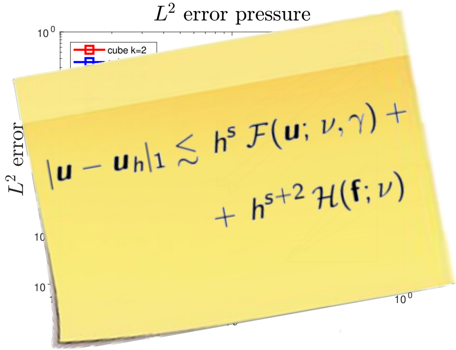
Example 3: Benchmark problem, case p_2





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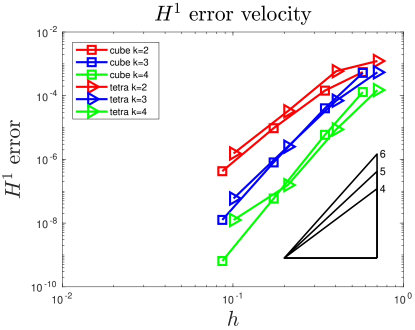
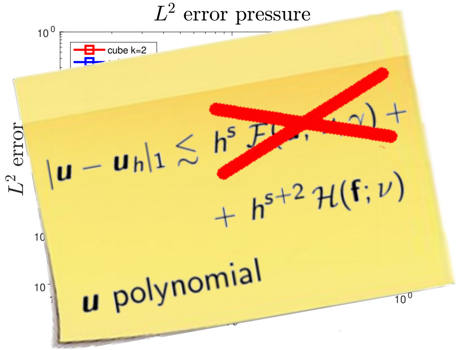
Example 3: Benchmark problem, case p_2





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Conclusions

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We presented Virtual Element approach

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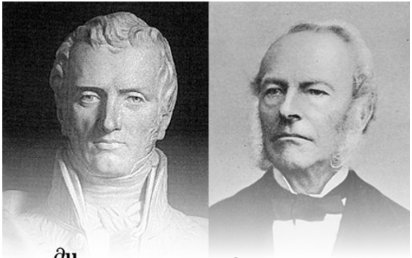
- for Stokes and Navier-Stokes problems 2d/3d



$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} = -\nabla p + \mathbf{g}.$$

Conclusions

We presented Virtual Element approach

- for Stokes and Navier-Stokes problems 2d/3d
- div-free property

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 \lesssim h^s \, \mathcal{F}(\boldsymbol{u};\, \nu, \gamma) + h^{s+2} \, \mathcal{H}(\mathbf{f}; \nu)$$

H^1 error velocity

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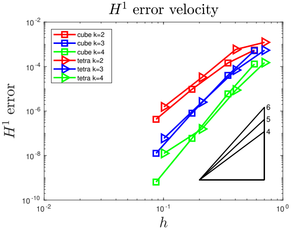

Related Pressure Robust and/or Div-free works

- L. Beirão da Veiga, C. Lovadina and G. Vacca, Virtual Elements for the Navier-Stokes problem on polygonal meshes, SIAM J. Numer. Anal, 2018;
- L. Beirão da Veiga, F. Dassi and G. Vacca, The Stokes complex for Virtual Elements in three dimensions, Math. Models Methods Appl. Sci., 2020;
- G. Wang, L. Mu, Y. Wang and Y. He, A pressure-robust virtual element method for the Stokes problem, Comput. Methods Appl. Mech. Eng., 2021;
- L. Beirão da Veiga, F. Dassi, D. A. Di Pietro and J. Droniou, Arbitrary-order pressure-robust DDR and VEM methods for the Stokes problem on polyhedral meshes, Comput. Methods Appl. Mech. Eng., 2022;
- D. Frerichs and C. Merdon, Divergence-preserving reconstruction on polygons and a really pressure robust virtual element method for the Stokes problem, IMA J. Numer. Anal., 2022;
- ...


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